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The spectrum of semi-Cayley graphs over abelian groups ☆

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ABSTRACT

In this paper, a formula of the spectrum of semi-Cayley graphs over finite abelian groups will be given. In particular, the spectrum of Cayley graphs over dihedral groups and dicyclic groups will be given, respectively.

1. Introduction

The concept of Cayley graphs was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. Let G be a group and H be a subset of G such that $e \notin H$. The *Cayley digraph* $Cay(G, H)$ over G with respect to H is a graph with vertex set $V = G$ and edge set $E = \{(x, y)|x, y \in G, yx^{-1} \in H\}$. A Cayley digraph $Cay(G, H)$ satisfying $H^{-1} = H$ is called *a Cayley graph*. Equivalently, a Cayley graph may be defined as a graph $\Gamma = (V, E)$ which admits an automorphism group acting regularly on the vertex set V. A graph is said to be a *semi-Cayley graph* over

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a group G if it admits G as a semiregular automorphism group with two orbits (of equal size). Resmini and Jungnickel gave the structure representation of semi-Cayley graphs in [11]. Let R, S, T be subsets of a group G such that $R = R^{-1}$, $S = S^{-1}$ and $e \notin R \cup S$. Define the undirected graph $SC(G; R, S, T)$ to have vertex set $G \times \{0, 1\}$, and with vertices (h, i), (g, j) adjacent if and only if one of the following three possibilities occurs:

(1) $i = j = 0$ and $gh^{-1} \in R$;
(2) $i = j = 1$ and $gh^{-1} \in S$;
(3) $i = 0$, $j = 1$ and $gh^{-1} \in T$.

Then $SC(G; R, S, T)$ is a semi-Cayley graph. Conversely, every semi-Cayley graph can be obtained in this way [11]. Investigation of semi-Cayley graphs is part of a larger project which aims at obtaining a deeper understanding of various classes of symmetric graphs [9,10].

Let Γ be a graph with vertices labeled as $0, 1, \ldots, n-1$. The adjacent matrix $A(\Gamma)$ of Γ is an $n \times n$ matrix with (i, j)-entry equals to 1 if vertices i and j are adjacent and 0 otherwise. The *spectrum* of a graph Γ is the set of numbers which are eigenvalues of $A(\Gamma)$, together with their multiplicities. We shall usually refer to the eigenvalues of $A(\Gamma)$ as the eigenvalues of Γ. It is known that numerous proofs in graph theory depend on the spectrum of graphs and the spectrum of a graph is its one of the most important algebraic invariants. The relationships between the algebraic properties of these eigenvalues and the usual properties of graphs have been studied intensively [1,3,5,6]. In particular, the spectrum of circulant digraphs was given by [4]. In [2], Babai derived an expression for the spectrum of Cayley graphs in terms of irreducible characters of groups. Xu [13] obtained a formula of the spectrum of Cayley graphs over finite abelian groups by an explicit expression of primitive roots of unity.

The aim of this paper is to study the spectrum of semi-Cayley graphs over finite abelian groups. We derive a formula of the spectrum of semi-Cayley graphs over finite abelian groups. As an application of our main result, we give a method to construct integral graphs. In particular, we obtain an explicit expression for the spectrum of Cayley graphs over two non-abelian groups (dihedral groups and dicyclic groups). The main result (Theorem 3.2) of this paper is

Theorem. Let $\Gamma = SC(G; R, S, T)$ be a semi-Cayley graph over a finite abelian group $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$. Then Γ has eigenvalues:

$$\frac{\lambda^R_{r_1\cdots r_t} + \lambda^S_{r_1\cdots r_t} \pm \sqrt{\left(\lambda^R_{r_1\cdots r_t} - \lambda^S_{r_1\cdots r_t}\right)^2 + 4\left|\lambda^T_{r_1\cdots r_t}\right|^2}}{2},$$

$r_j = 0, 1, \ldots, n_j - 1$, $j = 1, 2, \ldots, t$, where $\lambda^R_{r_1\cdots r_t}$, $\lambda^S_{r_1\cdots r_t}$ and $\lambda^T_{r_1\cdots r_t}$ are the eigenvalues of $Cay(G, R)$, $Cay(G, S)$ and $Cay(G, T)$, respectively.

2. Preliminary

Let $\mathbb{F}$ be a number field. As usual, denote by $\mathbb{F}^{m\times n}$ the set of $m \times n$ matrices for some positive integers m, n. An $n \times n$ matrix $C = (c_{ij})$ is said to be a *circulant matrix* if its entries satisfy $c_{ij} = c_{0(j-i)}$ for $i, j = 0, 1, \ldots, n-1$, where the subscripts are reduced modulo n. A Cayley digraph $Cay(G, H)$ is called a *circulant digraph* if $G = \mathbb{Z}_n$. It is known that a graph is a circulant digraph if and only if its adjacent matrix is a circulant matrix. Let W_n be the $n \times n$ circulant matrix whose first row is $(0 \quad 1 \quad 0 \quad \cdots \quad 0)$. It is known that if C is a circulant matrix whose first row is $\left(c_{00} \quad c_{01} \quad \cdots \quad c_{0(n-1)}\right)$, then $C = \sum_{i=0}^{n-1} c_{0i} W_n^i$ (where W_n^i is the ith power of the matrix W_n).

The *Kronecker product* $A \otimes B$ of two matrices A and B is the matrix obtained by replacing the (i, j)-entry a_{ij} of A by $a_{ij}B$, for all i and j. The Kronecker product has the following properties.

Lemma 2.1 [8, pp. 243–244]. *Let $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{p\times q}$, $C \in \mathbb{F}^{n\times k}$ and $D \in \mathbb{F}^{q\times r}$. Then*

$$(A \otimes B)^T = A^T \otimes B^T \text{ and } (A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Lemma 2.2 [8, pp. 243]. *Let $A, B \in \mathbb{F}^{m\times n}$ and $C \in \mathbb{F}^{p\times q}$. Then*

$$(A+B)\otimes C = A\otimes C + B\otimes C \quad \text{and} \quad C\otimes(A+B) = C\otimes A + C\otimes B.$$

The following results will be used in the sequel.

Lemma 2.3 [8]. *Let $A, B, C, D \in \mathbb{F}^{n\times n}$ with $|A| \neq 0$ and $AC = CA$. Then*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

Lemma 2.3 can be generalized to general case for matrix A.

Lemma 2.4. *Let $A, B, C, D \in \mathbb{F}^{n\times n}$ with $AC = CA$. Then*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

Proof. Let λ_0 be the smallest eigenvalue of A. For any $x < \lambda_0$, we have $|-xI_n + A| \neq 0$. Now $AC = CA$ implies that $(-xI_n + A)C = C(-xI_n + A)$. By Lemma 2.3 we have

$$\begin{vmatrix} -xI_n + A & B \\ C & D \end{vmatrix} = |(-xI_n + A)D - CB|. \quad (*)$$

Note that both sides of $(*)$ are polynomials of x with degree at most n. Then $(*)$ holds for all $x \in \mathbb{R}$. Especially for $x = 0$, we have

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|,$$

as required. □

Lemma 2.5 [8, pp. 251]. *Let A_i be a matrix of order n_i, $i = 1, 2, \ldots, k$ and*

$$f(x_1, x_2, \ldots, x_k) = \sum_{i_1,\ldots,i_k} a_{i_1\cdots i_k} x_1^{i_1} \cdots x_k^{i_k}.$$

Suppose that each A_i has eigenvalues λ_{it_i} $(1 \leqslant t_i \leqslant n_i)$. Then

$$f(A_1, \ldots, A_k) = \sum_{i_1,\ldots,i_k} a_{i_1\cdots i_k} A_1^{i_1} \otimes \cdots \otimes A_k^{i_k}$$

has eigenvalues:

$$f(\lambda_{1t_1}, \ldots, \lambda_{kt_k}) = \sum_{i_1,\ldots,i_k} a_{i_1\cdots i_k} \lambda_{1t_1}^{i_1} \cdots \lambda_{kt_k}^{i_k}.$$

Recall that $Cay(G, H)$ is undirected if and only if $H^{-1} = H$. The spectrum of undirected Cayley graphs over finite abelian groups was determined in [13]. With a similar argument, one can obtain the spectrum of any Cayley digraph over a finite abelian group. Let ω_n be the nth primitive unity root.

Lemma 2.6. *Let $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$ and H be a subset of $G \setminus \{e\}$. Then the adjacent matrix of Cayley graph $Cay(G, H)$ is*

$$A(X) = \sum_{(i_1,\ldots,i_t)\in H} W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t}$$

and the eigenvalues of $Cay(G, H)$ are

$$\lambda_{r_1\ldots r_t} = \sum_{(i_1,\ldots,i_t)\in H} \omega_{n_1}^{i_1 r_1}\cdots\omega_{n_t}^{i_t r_t}, \quad r_j = 0,1,\ldots,n_j-1,\ j=1,2,\ldots,t.$$

The reader is referred to [4,12] for all the notation and terminology not defined in this paper.

3. The spectrum of semi-Cayley graphs over finite abelian groups

In this section, we shall study the spectrum of semi-Cayley graphs over finite abelian groups and give a formula of the spectrum. We first give some information about the adjacent matrix of a semi-Cayley graph. All the subscripts in next lemma are reduced modulo n.

Lemma 3.1. *Let $\Gamma = SC(G;R,S,T)$ be a semi-Cayley graph over a finite abelian group $G = \mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_t}$, A, B, C and D be the adjacent matrices of $Cay(G,R)$, $Cay(G,T)$, $Cay(G,S)$ and Γ, respectively. Then*

(1) $D = \begin{pmatrix} A & B \\ B^T & C\end{pmatrix}$, *where* $A = \sum_{(i_1,\ldots,i_t)\in R} W_{n_1}^{i_1}\otimes\cdots\otimes W_{n_t}^{i_t}$, $B = \sum_{(i_1,\ldots,i_t)\in T} W_{n_1}^{i_1}\otimes\cdots\otimes W_{n_t}^{i_t}$ *and* $C = \sum_{(i_1,\ldots,i_t)\in S} W_{n_1}^{i_1}\otimes\cdots\otimes W_{n_t}^{i_t}$;

(2) $AC = \sum_{\substack{(i_1,\ldots,i_t)\in R\\(j_1,\ldots,j_t)\in S}} W_{n_1}^{i_1+j_1}\otimes\cdots\otimes W_{n_t}^{i_t+j_t}$;

(3) $B^T = \sum_{(i_1,\ldots,i_t)\in T} W_{n_1}^{n_1-i_1}\otimes\cdots\otimes W_{n_t}^{n_t-i_t}$ *and* $B^TB = \sum_{\substack{(i_1,\ldots,i_t)\in T\\(j_1,\ldots,j_t)\in T}} W_{n_1}^{i_1+n_1-j_1}\otimes\cdots\otimes W_{n_t}^{i_t+n_t-j_t}$;

(4) $AB^T = B^TA$.

Proof

(1) Let $G = \{g_0, g_1, \ldots, g_{n-1}\}$. By re-ordering the vertices of Γ (if necessary), we can assign D to be a block matrix as follows:

$$D = \begin{pmatrix} A & D_1 \\ D_2 & C\end{pmatrix}.$$

To prove that $D_1 = B$, let $D_1 = (\bar{d}_{ij})$ and $B = (b_{ij})$. Then

$$\begin{aligned}\bar{d}_{ij} = 1 &\Leftrightarrow ((g_i,0),(g_j,1)) \in E(\Gamma) \text{ for } g_i, g_j \in G \Leftrightarrow g_jg_i^{-1} \in T\\ &\Leftrightarrow (g_i,g_j) \in E(Cay(G,T)) \Leftrightarrow b_{ij} = 1.\end{aligned}$$

It follows from the fact that D_1 and B are (0, 1)-matrices that $\bar{d}_{ij} = b_{ij}$ for all $i,j = 0,1,\ldots,n-1$. Hence $D_1 = B$. Next we show that $D_2 = B^T$. Let $D_2 = \left(d'_{ij}\right)$. Then

$$\begin{aligned}d'_{ij} = 1 &\Leftrightarrow ((g_i,1),(g_j,0)) \in E(\Gamma) \text{ for } g_i, g_j \in G \Leftrightarrow g_ig_j^{-1} \in T\\ &\Leftrightarrow (g_j,g_i) \in E(Cay(G,T)) \Leftrightarrow b_{ji} = 1.\end{aligned}$$

Since D_2 and B^T are (0, 1)-matrices, we have $d'_{ij} = b_{ji}$ for all $i,j = 0,1,\ldots,n-1$. Hence $D_2 = B^T$. By Lemma 2.6, we get that A, B and C are of the form indicated in the lemma.

(2) It follows from Lemmas 2.1 and 2.2 immediately.

(3) Note that $\left(W_n^i\right)^T = W_n^{n-i}$. By Lemma 2.1, we have

$$\begin{aligned}B^T &= \sum_{(j_1,\ldots,j_t)\in T}\left(W_{n_1}^{j_1}\otimes\cdots\otimes W_{n_t}^{j_t}\right)^T\\ &= \sum_{(j_1,\ldots,j_t)\in T}\left(W_{n_1}^{j_1}\right)^T\otimes\cdots\otimes(W_{n_t}^{j_t})^T\end{aligned}$$

$$= \sum_{(j_1,\ldots,j_t)\in T} W_{n_1}^{n_1-j_1} \otimes \cdots \otimes W_{n_t}^{n_t-j_t}.$$

The remainder follows from Lemmas 2.1 and 2.2.

(4) By Lemmas 2.1, 2.2 and (3), we have

$$AB^T = \sum_{\substack{(i_1,\ldots,i_t)\in R\\(j_1,\ldots,j_t)\in T}} \left(W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t}\right)\left(W_{n_1}^{n_1-j_1} \otimes \cdots \otimes W_{n_t}^{n_t-j_t}\right)$$
$$= \sum_{\substack{(i_1,\ldots,i_t)\in R\\(j_1,\ldots,j_t)\in T}} W_{n_1}^{i_1+n_1-j_1} \otimes \cdots \otimes W_{n_t}^{i_t+n_t-j_t}$$

and

$$B^TA = \sum_{\substack{(i_1,\ldots,i_t)\in R\\(j_1,\ldots,j_t)\in T}} \left(W_{n_1}^{n_1-j_1} \otimes \cdots \otimes W_{n_t}^{n_t-j_t}\right)\left(W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t}\right)$$
$$= \sum_{\substack{(i_1,\ldots,i_t)\in R\\(j_1,\ldots,j_t)\in T}} W_{n_1}^{i_1+n_1-j_1} \otimes \cdots \otimes W_{n_t}^{i_t+n_t-j_t}.$$

Hence $AB^T = B^TA$, as required. □

Remark. For semi-Cayley graph $SC(G; R, S, T)$, since $R^{-1} = R$ and $S^{-1} = S$, $Cay(G, R)$ and $Cay(G, S)$ are undirected. So the adjacent matrices of $Cay(G, R)$ and $Cay(G, S)$ are symmetric and their spectrum are real. However, T may not be inverse closed, hence the spectrum of $Cay(G, T)$ may not be real.

Now we are ready to give the spectrum of semi-Cayley graphs over finite abelian groups. Let a be a complex number, denote by a^* the conjugate complex number of a. It is known that $\left(\omega_n^i\right)^* = \omega_n^{n-i}$. Let $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$ and H be a subset of G such that $e \notin H$. For $0 \leqslant r_j \leqslant n_j - 1$, $1 \leqslant j \leqslant t$, let $\lambda_{r_1\cdots r_t}^H = \sum_{(i_1,\ldots,i_t)\in H} \omega_{n_1}^{r_1 i_1} \cdots \omega_{n_t}^{r_t i_t}$. Note that if $H = \emptyset$, then $\lambda_{r_1\cdots r_t}^H = 0$.

Theorem 3.2. *Let $\Gamma = SC(G; R, S, T)$ be a semi-Cayley graph over a finite abelian group $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$. Then Γ has eigenvalues:*

$$\frac{\lambda_{r_1\cdots r_t}^R + \lambda_{r_1\cdots r_t}^S \pm \sqrt{\left(\lambda_{r_1\cdots r_t}^R - \lambda_{r_1\cdots r_t}^S\right)^2 + 4\left|\lambda_{r_1\cdots r_t}^T\right|^2}}{2},$$

$r_j = 0, 1, \ldots, n_j - 1$, $j = 1, 2, \ldots, t$, where $\lambda_{r_1\cdots r_t}^R$, $\lambda_{r_1\cdots r_t}^S$ and $\lambda_{r_1\cdots r_t}^T$ are the eigenvalues of $Cay(G, R)$, $Cay(G, S)$ and $Cay(G, T)$, respectively.

Proof. Let A, B, C and D be the adjacent matrices of $Cay(G, R)$, $Cay(G, T)$, $Cay(G, S)$ and Γ, respectively. Then by Lemma 3.1

$$D = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where

$$A = \sum_{(i_1,\ldots,i_t)\in R} W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t}, \quad B = \sum_{(i_1,\ldots,i_t)\in T} W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t},$$
$$C = \sum_{(i_1,\ldots,i_t)\in S} W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t}.$$

It follows from Lemma 2.6 that $\lambda^R_{r_1\cdots r_t}$, $\lambda^S_{r_1\cdots r_t}$ and $\lambda^T_{r_1\cdots r_t}$ are the eigenvalue of $Cay(G,R)$, $Cay(G,S)$ and $Cay(G,T)$, respectively. Suppose that $|G| = n$. Let λ be an eigenvalue of D. Then

$$0 = |D - \lambda I_{2n}| = \begin{vmatrix} A - \lambda I_n & B \\ B^T & C - \lambda I_n \end{vmatrix}.$$

Since $AB^T = B^T A$ by Lemma 3.1, we have $(A - \lambda I_n)B^T = B^T(A - \lambda I_n)$. It follows from Lemma 2.4 that

$$0 = \left|(A - \lambda I_n)(C - \lambda I_n) - B^T B\right| = \left|\lambda^2 I_n - \lambda A - \lambda C + AC - B^T B\right|. \tag{1}$$

Let $E = \lambda^2 I_n - \lambda A - \lambda C + AC - B^T B$. Then

$$\begin{aligned} E = {} & \lambda^2 I_n - \lambda \sum_{(i_1,\ldots,i_t)\in R} W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t} - \lambda \sum_{(i_1,\ldots,i_t)\in S} W_{n_1}^{i_1} \otimes \cdots \otimes W_{n_t}^{i_t} \\ & + \sum_{\substack{(i_1,\ldots,i_t)\in R \\ (j_1,\ldots,j_t)\in S}} W_{n_1}^{i_1+j_1} \otimes \cdots \otimes W_{n_t}^{i_t+j_t} - \sum_{\substack{(i_1,\ldots,i_t)\in T \\ (j_1,\ldots,j_t)\in T}} W_{n_1}^{i_1+n_1-j_1} \otimes \cdots \otimes W_{n_t}^{i_t+n_t-j_t}. \end{aligned}$$

Note that for any $i = 0, 1, \ldots, n-1$, W_n^i has eigenvalues ω_n^{ir}, $r = 0, 1, \ldots, n-1$. By Lemma 2.5, E has eigenvalues

$$\begin{aligned} \lambda_{r_1\cdots r_t} = {} & \lambda^2 - \lambda \sum_{(i_1,\ldots,i_t)\in R} \omega_{n_1}^{i_1 r_1} \cdots \omega_{n_t}^{i_t r_t} - \lambda \sum_{(i_1,\ldots,i_t)\in S} \omega_{n_1}^{i_1 r_1} \cdots \omega_{n_t}^{i_t r_t} \\ & + \sum_{\substack{(i_1,\ldots,i_t)\in R \\ (j_1,\ldots,j_t)\in S}} \omega_{n_1}^{(i_1+j_1)r_1} \cdots \omega_{n_t}^{(i_t+j_t)r_t} - \sum_{\substack{(i_1,\ldots,i_t)\in T \\ (j_1,\ldots,j_t)\in T}} \omega_{n_1}^{(i_1+n_1-j_1)r_1} \cdots \omega_{n_t}^{(i_t+n_t-j_t)r_t} \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \sum_{\substack{(i_1,\ldots,i_t)\in R \\ (j_1,\ldots,j_t)\in S}} \left(\omega_{n_1}^{i_1 r_1} \cdots \omega_{n_t}^{i_t r_t}\right)\left(\omega_{n_1}^{j_1 r_1} \cdots \omega_{n_t}^{j_t r_t}\right) \\ & - \sum_{\substack{(i_1,\ldots,i_t)\in T \\ (j_1,\ldots,j_t)\in T}} \left(\omega_{n_1}^{i_1 r_1} \cdots \omega_{n_t}^{i_t r_t}\right)\left(\omega_{n_1}^{(n_1-j_1)r_1} \cdots \omega_{n_t}^{(n_t-j_t)r_t}\right) \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \left(\sum_{(i_1,\ldots,i_t)\in R} \omega_{n_1}^{i_1 r_1} \cdots \omega_{n_t}^{i_t r_t}\right)\left(\sum_{(j_1,\ldots,j_t)\in S} \omega_{n_1}^{j_1 r_1} \cdots \omega_{n_t}^{j_t r_t}\right) \\ & - \left(\sum_{(i_1,\ldots,i_t)\in T} \omega_{n_1}^{i_1 r_1} \cdots \omega_{n_t}^{i_t r_t}\right)\left(\sum_{(j_1,\ldots,j_t)\in T} \omega_{n_1}^{(n_1-j_1)r_1} \cdots \omega_{n_t}^{(n_t-j_t)r_t}\right) \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \lambda^R_{r_1\cdots r_t}\lambda^S_{r_1\cdots r_t} - \lambda^T_{r_1\cdots r_t} \sum_{(j_1,\ldots,j_t)\in T} \left(\omega_{n_1}^{j_1 r_1}\right)^* \cdots \left(\omega_{n_t}^{j_t r_t}\right)^* \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \lambda^R_{r_1\cdots r_t}\lambda^S_{r_1\cdots r_t} - \lambda^T_{r_1\cdots r_t} \sum_{(j_1,\ldots,j_t)\in T} \left(\omega_{n_1}^{j_1 r_1} \cdots \omega_{n_t}^{j_t r_t}\right)^* \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \lambda^R_{r_1\cdots r_t}\lambda^S_{r_1\cdots r_t} - \lambda^T_{r_1\cdots r_t} \left(\sum_{(j_1,\ldots,j_t)\in T} \omega_{n_1}^{j_1 r_1} \cdots \omega_{n_t}^{j_t r_t}\right)^* \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \lambda^R_{r_1\cdots r_t}\lambda^S_{r_1\cdots r_t} - \lambda^T_{r_1\cdots r_t} \left(\lambda^T_{r_1\cdots r_t}\right)^* \\ = {} & \lambda^2 - \lambda\lambda^R_{r_1\cdots r_t} - \lambda\lambda^S_{r_1\cdots r_t} + \lambda^R_{r_1\cdots r_t}\lambda^S_{r_1\cdots r_t} - \left|\lambda^T_{r_1\cdots r_t}\right|^2, \end{aligned}$$

where $r_j = 0, 1, \ldots, n_j - 1$, $j = 1, 2, \ldots, t$. Hence

$$0 = |D - \lambda I_{2n}| = \prod_{\substack{0 \leqslant r_1 \leqslant n_1 - 1 \\ \cdots \\ 0 \leqslant r_t \leqslant n_t - 1}} \lambda_{r_1 \cdots r_t}.$$

Let $\lambda_{r_1 \ldots r_t} = 0$. Then we have

$$\begin{aligned} \lambda &= \frac{\lambda^R_{r_1\cdots r_t} + \lambda^S_{r_1\cdots r_t} \pm \sqrt{\left(\lambda^R_{r_1\cdots r_t} + \lambda^S_{r_1\cdots r_t}\right)^2 - 4\left(\lambda^R_{r_1\cdots r_t}\lambda^S_{r_1\cdots r_t} - \left|\lambda^T_{r_1\cdots r_t}\right|^2\right)}}{2} \\ &= \frac{\lambda^R_{r_1\cdots r_t} + \lambda^S_{r_1\cdots r_t} \pm \sqrt{\left(\lambda^R_{r_1\cdots r_t} - \lambda^S_{r_1\cdots r_t}\right)^2 + 4\left|\lambda^T_{r_1\cdots r_t}\right|^2}}{2}. \end{aligned}$$

Hence D has eigenvalues

$$\frac{\lambda^R_{r_1\cdots r_t} + \lambda^S_{r_1\cdots r_t} \pm \sqrt{\left(\lambda^R_{r_1\cdots r_t} - \lambda^S_{r_1\cdots r_t}\right)^2 + 4\left|\lambda^T_{r_1\cdots r_t}\right|^2}}{2},$$

$r_j = 0, 1, \ldots, n_j - 1$, $j = 1, 2, \ldots, t$, as required. □

As a direct consequence of Theorem 3.2, we have

Corollary 3.3. *Let $\Gamma = SC(\mathbb{Z}_n; R, R, T)$ be a semi-Cayley graph. Then Γ has eigenvalues*:

$$\lambda^R_r \pm \left|\lambda^T_r\right|, \quad r = 0, 1, \ldots, n-1,$$

where $\lambda^R_r = \sum_{i\in R} \omega_n^{ir}$ and $\lambda^T_r = \sum_{i\in T} \omega_n^{ir}$, $r = 0, 1, \ldots, n-1$ are the eigenvalues of $Cay(\mathbb{Z}_n, R)$ and $Cay(\mathbb{Z}_n, T)$, respectively.

Example 3.4. Let $\Gamma = SC(\mathbb{Z}_3; R, R, T)$ be a semi-Cayley graph with $R = \{1, 2\}$ and $T = \{2\}$. Then $Cay(\mathbb{Z}_3, R)$ has eigenvalues $\{-1, -1, 2\}$, $Cay(\mathbb{Z}_3, T)$ has eigenvalues $\{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\}$ and Γ has eigenvalues $\{-1 \pm 1, -1 \pm 1, 2 \pm 1\} = \{-2, -2, 0, 0, 1, 3\}$.

A graph is called an *integral graph* if it has an integral spectrum, i.e., all eigenvalues are integers. Xu [13] completely determined the integral Cayley graphs over finite abelian groups. The next corollary gives a method to construct integral graphs.

Corollary 3.5. *Let $\Gamma = SC(G; R, R, T)$ be a semi-Cayley graph over $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$. If $Cay(G, R)$ and $Cay(G, T)$ are integral graphs, then Γ is an integral graph.*

Proof. By Theorem 3.2, Γ has eigenvalues $\lambda^R_{r_1\cdots r_t} \pm |\lambda^T_{r_1\cdots r_t}|, r_j = 0, 1, \ldots, n_j - 1, j = 1, 2, \ldots, t$, where $\lambda^R_{r_1\cdots r_t}$ and $\lambda^T_{r_1\cdots r_t}$ are eigenvalues of $Cay(G, R)$ and $Cay(G, T)$, respectively. The assertion of corollary follows immediately. □

Example 3.4 shows that the converse of Corollary 3.5 is not true. As a consequence of Theorem 3.2, we have the following result which was obtained in [14].

Corollary 3.6. *Let $\Gamma = SC(G; \emptyset, \emptyset, T)$ be a semi-Cayley graph over $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$. Then Γ has eigenvalues*:

$$\pm\left|\lambda^T_{r_1\cdots r_t}\right|, \quad r_j = 0, 1, \ldots, n_j - 1, \; j = 1, 2, \ldots, t,$$

where $\lambda^T_{r_1\cdots r_t}$ are the eigenvalues of $Cay(G, T)$.

4. Some special cases

In this section, we shall give the spectrum of Cayley graphs over dihedral groups D_n and dicyclic groups DC_{2n}, respectively.

Let $D_n = \langle a, x | a^n = x^2 = 1, x^{-1}ax = a^{-1} \rangle$ be the dihedral group, $H \subseteq D_n \setminus \{e\}$ with $H^{-1} = H$ and $H' = \{i | a^i \in H\}$. Denote by Γ_1 and Γ_2 the subgraphs induced by $\{a^i | 0 \leqslant i \leqslant n-1\}$ and $\{a^i x | 0 \leqslant i \leqslant n-1\}$ in $Cay(D_n, H)$, respectively.

Lemma 4.1 [7]. *$\Gamma_1 \cong \Gamma_2 \cong Cay(\mathbb{Z}_n, H')$ and $Cay(D_n, H)$ can be decomposed into two subgraphs Γ_1, Γ_2 together with a class of perfect matchings joining them.*

In view of [7], for any $a^i x \in H$, the set of edges of color $a^i x$ is a perfect matching of $Cay(D_n, H)$ joining Γ_1 and Γ_2. To determine the spectrum of $Cay(D_n, H)$, we first give a structure representation of $Cay(D_n, H)$ by semi-Cayley graphs. Recall that the union of two graphs X and Y is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y)$. Furthermore, if $V(X) \cap V(Y) = \emptyset$, we call the union of X and Y is disjoint and denote it by $X + Y$.

Lemma 4.2

(1) *If $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} = \emptyset$, then $Cay(D_n, H) \cong SC(\mathbb{Z}_n; H', H', \emptyset)$;*
(2) *If $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} \neq \emptyset$, let $a^{i_0} x \in H$, then $Cay(D_n, H) \cong SC\left(\mathbb{Z}_n; H', H', T\right)$, where $T = \{i | a^{i_0 - i} x \in H\}$.*

Proof

(1) Since $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} = \emptyset$, it follows that $Cay(D_n, H) = \Gamma_1 + \Gamma_2$ and $\Gamma_1 \cong \Gamma_2 \cong Cay\left(\mathbb{Z}_n, H'\right)$ by Lemma 4.1. Hence $Cay(D_n, H) \cong SC\left(\mathbb{Z}_n; H', H', \emptyset\right)$.
(2) Let $\Gamma = SC\left(\mathbb{Z}_n; H', H', T\right)$. Define a mapping $\phi : Cay(D_n, H) \to \Gamma$ by $\phi(a^i) = (i, 0)$ and $\phi(a^i x) = (i_0 - i, 1)$. It is clear that ϕ is a bijection from $V(Cay(D_n, H))$ to $V(\Gamma)$. To prove that ϕ is an isomorphism, let $b, c \in V(Cay(D_n, H))$. If $b = a^i$ and $c = a^j$ for some $a^i, a^j \in D_n$, then

$$\begin{aligned}(b, c) \in E(Cay(D_n, H)) &\Leftrightarrow a^j (a^i)^{-1} = a^{j-i} \in H \Leftrightarrow j - i \in H' \\ &\Leftrightarrow ((i, 0), (j, 0)) \in E(\Gamma) \\ &\Leftrightarrow (\phi(b), \phi(c)) \in E(\Gamma).\end{aligned}$$

If $b = a^i$ and $c = a^j x$ for some $a^i, a^j x \in D_n$, then

$$\begin{aligned}(b, c) \in E(Cay(D_n, H)) &\Leftrightarrow a^j x (a^i)^{-1} = a^{j+i} x \in H \Leftrightarrow i_0 - j - i \in T \\ &\Leftrightarrow ((i, 0), (i_0 - j, 1)) \in E(\Gamma) \\ &\Leftrightarrow (\phi(b), \phi(c)) \in E(\Gamma).\end{aligned}$$

If $b = a^i x$ and $c = a^j x$ for some $a^i x, a^j x \in D_n$, then

$$\begin{aligned}(b, c) \in E(Cay(D_n, H)) &\Leftrightarrow a^j x (a^i x)^{-1} = a^{j-i} \in H \Leftrightarrow a^{i-j} \in H \Leftrightarrow i - j \in H' \\ &\Leftrightarrow ((i_0 - i, 1), (i_0 - j, 1)) \in E(\Gamma) \\ &\Leftrightarrow (\phi(b), \phi(c)) \in E(\Gamma).\end{aligned}$$

Therefore ϕ is an isomorphism, as required. □

Now we may give the spectrum of $Cay(D_n, H)$.

Theorem 4.3. *Let $Cay(D_n, H)$ be a Cayley graph.*

(1) *If $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} = \emptyset$, then $Cay(D_n, H)$ has eigenvalues $\sum_{a^i \in H} \omega_n^{ir}$ with multiplicity 2, $r = 0, 1, \ldots, n-1$;*
(2) *If $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} \neq \emptyset$, let $a^{i_0}x \in H$, then $Cay(D_n, H)$ has eigenvalues*

$$\sum_{a^i \in H} \omega_n^{ir} \pm \left| \sum_{a^i x \in H} \omega_n^{(i_0 - i)r} \right|, \quad r = 0, 1, \ldots, n-1.$$

Proof

(1) If $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} = \emptyset$, then $Cay(D_n, H) \cong SC\left(\mathbb{Z}_n; H', H', \emptyset\right)$ by Lemma 4.2. By Corollary 3.3, $Cay(D_n, H)$ has eigenvalues $\sum_{a^i \in H} \omega_n^{ir}$ with multiplicity 2, $r = 0, 1, \ldots, n-1$.
(2) If $H \cap \{a^i x | 0 \leqslant i \leqslant n-1\} \neq \emptyset$, then by Lemma 4.2, $Cay(D_n, H) \cong SC(\mathbb{Z}_n; H', H', T)$, where $T = \{i | a^{i_0 - i}x \in H\}$. By Corollary 3.3, $Cay(D_n, H)$ has eigenvalues

$$\begin{aligned} \sum_{i \in H'} \omega_n^{ir} \pm \left| \sum_{i \in T} \omega_n^{ir} \right| &= \sum_{a^i \in H} \omega_n^{ir} \pm \left| \sum_{a^{i_0 - i}x \in H} \omega_n^{ir} \right| \\ &= \sum_{a^i \in H} \omega_n^{ir} \pm \left| \sum_{a^i x \in H} \omega_n^{(i_0 - i)r} \right|, \end{aligned}$$

$r = 0, 1, \ldots, n-1$, as required. □

Next we consider the spectrum of Cayley graphs over dicyclic groups. Let $DC_{2n} = \langle a, x | a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ be the dicyclic group, $H \subseteq DC_{2n} \setminus \{e\}$ with $H^{-1} = H$ and $H' = \{i | a^i \in H\}$. Denote by Γ_1 and Γ_2 the subgraphs induced by $\{a^i | 0 \leqslant i \leqslant 2n-1\}$ and $\{a^i x | 0 \leqslant i \leqslant 2n-1\}$ in $Cay(DC_{2n}, H)$, respectively. By a similar argument as in the proof of Lemma 4.1, we get the following result.

Lemma 4.4. *$\Gamma_1 \cong \Gamma_2 \cong Cay(\mathbb{Z}_{2n}, H')$ and $Cay(DC_{2n}, H)$ can be decomposed into two subgraphs Γ_1, Γ_2 together with a class of perfect matchings joining them.*

To determine the spectrum of $Cay(DC_{2n}, H)$, we give a structure representation of $Cay(DC_{2n}, H)$ by semi-Cayley graphs.

Lemma 4.5

(1) *If $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} = \emptyset$, then $Cay(DC_{2n}, H) \cong SC\left(\mathbb{Z}_{2n}; H', H', \emptyset\right)$;*
(2) *If $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} \neq \emptyset$, let $a^{i_0}x \in H$, then $Cay(DC_{2n}, H) \cong SC\left(\mathbb{Z}_{2n}; H', H', T\right)$, where $T = \{i | a^{i_0 - i}x \in H\}$.*

Proof

(1) Since $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} = \emptyset$, we have that $Cay(DC_{2n}, H) = \Gamma_1 + \Gamma_2$ and $\Gamma_1 \cong \Gamma_2 \cong Cay\left(\mathbb{Z}_{2n}, H'\right)$ by Lemma 4.4. So $Cay(DC_{2n}, H) \cong SC\left(\mathbb{Z}_{2n}; H', H', \emptyset\right)$.
(2) Let $\Gamma = SC\left(\mathbb{Z}_{2n}; H', H', T\right)$. Define a map $\phi : Cay(DC_{2n}, H) \to \Gamma$ by $\phi(a^i) = (i, 0)$ and $\phi(a^i x) = (i_0 - i, 1)$. Then obviously ϕ is a bijection from $V(Cay(DC_{2n}, H))$ to $V(\Gamma)$. With a similar argument of the proof in Lemma 4.2, we can derive that ϕ is an isomorphism. □

Now we may give the spectrum of the Cayley graphs over dicyclic group.

Theorem 4.6. *Let $Cay(DC_{2n}, H)$ be a Cayley graph.*

(1) *If $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} = \emptyset$, then $Cay(DC_{2n}, H)$ has eigenvalues $\sum_{a^i \in H} \omega_n^{ir}$ with multiplicity 2, $r = 0, 1, \ldots, 2n-1$;*
(2) *If $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} \neq \emptyset$, let $a^{i_0} x \in H$, then $Cay(DC_{2n}, H)$ has eigenvalues*

$$\sum_{a^i \in H} \omega_n^{ir} \pm \left| \sum_{a^i x \in H} \omega_n^{(i_0 - i)r} \right|, \quad r = 0, 1, \ldots, 2n-1.$$

Proof

(1) If $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} = \emptyset$, then $Cay(DC_{2n}, H) \cong SC\left(\mathbb{Z}_{2n}; H', H', \emptyset\right)$ by Lemma 4.5. By Corollary 3.3, $Cay(DC_{2n}, H)$ has eigenvalues $\sum_{a^i \in H} \omega_n^{ir}$ with multiplicity 2, $r = 0, 1, \ldots, 2n-1$.
(2) If $H \cap \{a^i x | 0 \leqslant i \leqslant 2n-1\} \neq \emptyset$, then by Lemma 4.5, $Cay(DC_{2n}, H) \cong SC\left(\mathbb{Z}_{2n}; H', H', T\right)$, where $T = \{i | a^{i_0 - i} x \in H\}$. By Corollary 3.3, $Cay(DC_{2n}, H)$ has eigenvalues

$$\begin{aligned} \sum_{i \in H'} \omega_n^{ir} \pm \left| \sum_{i \in T} \omega_n^{ir} \right| &= \sum_{a^i \in H} \omega_n^{ir} \pm \left| \sum_{a^{i_0 - i} x \in H} \omega_n^{ir} \right| \\ &= \sum_{a^i \in H} \omega_n^{ir} \pm \left| \sum_{a^i x \in H} \omega_n^{(i_0 - i)r} \right|, \end{aligned}$$

$r = 0, 1, \ldots, 2n-1$, as required. □

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